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AC TRANSPORT CURRENT LOSS IN A COATED SUPERCONDUCTOR IN THE BEAN MODEL

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AC transport current loss in a coated superconductor in the Bean model

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Abstract

A new and straightforward calculation is made of the loss in a very thin superconducting strip of rectangular cross-section (e.g. the coating on a coated superconductor) carrying ac transport current in zero applied magnetic field, with a similar strip acting as the return path. The computation is made assuming only that the strip is composed of uniform material which obeys Maxwell's equations and the Bean model. An important consequence of the Bean model is the existence of a field-free region about the middle of the superconductor cross-section. The present loss calculation is novel in two respects: (1) It uses for the first time an actual computation of the shape of the field-free region rather than using qualitative assumptions, and (2) it uses a new approach for making the loss calculation, based on a rigorous solution of Maxwell's equations for this problem. The rigorous solution correctly treats the problem as three-dimensional, having a time-dependent charge on the surface of the superconductor, and having the electric field described by both a vector and a scalar potential. Loss computations are made for the ratio of peak current to critical current in the approximate range of one-half to one, where within this range the loss decreases by about two powers of 10. The most important result coming out of the present calculation (made for the case of a distant return path large compared with the conductor cross-section dimensions but small compared with the length of the conductor), is a confirmation of Norris's previously estimated loss expression which he obtained in a different way.

1. Introduction

Quite recently the transport-current loss in a "coated superconductor", i.e. a thick-film of high-temperature superconductor deposited on a buffer layer and substrate such as to have biaxial

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crystal orientation, has become of interest. Loss measurements are now available for both ion bean assisted deposition (IBAD) [1–4] and rolling assisted biaxially textured substrates (RABiTS) [5] processed material. In regard to dependence of the loss on peak transport current most of these measurements disagree with the theory various authors have developed for these materials, and it is still questionable as to whether the disagreement is due to errors in the theory or to non-uniformity in the measured samples. The purpose of the present analysis is to examine the correctness of the theory by (a) developing a new approach for making the loss calculation, where all the approximations are clearly stated, and (b) by assuming a "field-free region" where, for the first time, an actual calculation of its shape is used rather than a rough estimated shape. The field-free region arises from the use of a Bean model for the superconductor, and in the case of a transport current it exists at the center of the superconductor cross-section, which has no current density or field throughout a cycle of peak current less than the critical current.

In some sense coated superconductors are simply strips of rectangular cross-section having a very large aspect ratio. However, the term also implies a high-temperature superconducting material (usually YBCO) with biaxially oriented crystallites, having very low-angle grain boundaries and therefore having little "granularity". Consequently one might expect the Bean model to apply. Loss theory for a uniform superconducting strip carrying transport current was originally developed by Norris [6], who followed an approach used by London [7] for the case of a circular conductor. In Norris's approach the strip was imagined to be infinite in length and broken up into "fibres", for which the loss was computed using circuit theory, the Bean model, and a starting expression given in terms of the flux between fibres. Although the result was found to depend strongly on the field-free region Norris approximated the loss for a thin strip without actually computing this region. More recently Rhyner [8] has published a simplified loss calculation based on a vector potential formulation together with the Bean model but his calculation is limited to the special case of peak current equal to the critical current. Analytic com-

putations have also been made by several other authors, along with a number of numerical calculations (for example Amemiya et al. [9]). However, the problem is more complex than most authors indicate. In nearly every case tacit approximations are used and it is often difficult to know what the approximations are. Even the term Bean model means different things to different authors.

A special type of approach for estimating the loss in thin films has been given by Brandt and Indenbom [10] who argue that for a thin film the Bean model does not apply, and a similar approach to thin films was published by Zeldov et al. [11]. But if indeed these assumptions are correct for a thin film, it will be assumed here that in typical coated superconductors the coating is sufficiently thick (about one micrometer) that the Bean model does apply, except possibly for very small currents.

In the new approach to be developed it is assumed only that the superconductor is described by quasi-static Maxwell equations (with no true magnetization or electric polarization) plus the Bean model, and the first step in computing the transport-current loss is made by computing a precise vector and scalar potential that solves the Maxwell equations for the given geometry, before a "constitutive" relation (i.e. the Bean model) is introduced. This solution is much different than one might expect from a comparison with the applied magnetic-field case [12], mainly because of a surface charge which exists on the conductor when a transport current flows [13]. For this reason both a vector potential $\mathbf{A}$ and a scalar potential ϕ must be considered, and the electric field is obtained from $\mathbf{E} = -\dot{\mathbf{A}} - \nabla\phi$ where the dot indicates a time derivative. The flux density is $\mathbf{B} = \text{curl}\,\mathbf{A}$. In computing the potentials one must consider the problem to be three-dimensional, and furthermore that the standard integrals which give the potentials must be performed over a complete circuit. Consequently the potentials depend on the geometry of the return path for the current in a given conductor. The geometry considered here consists of a long straight conductor with a similar parallel conductor acting as the return path. The two conductors are separated a distance large compared with the dimensions of the conductor cross-sec-

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tion, but small compared with the length of the straight sections. It is sometimes assumed that this three-dimensional geometry can be replaced by an infinitely long single conductor. The latter is not true since it leads to an infinite term in the electric field, which can be removed only by also considering the return path.

The Bean model has two parts: (a) It assumes that the smallest electric field produces a fixed critical current density J_c in the direction of the electric field, and (b) if an electric field is applied and then reduced to zero the current density remains at J_c in its original direction (the current density vanishes only in a virgin material). The Bean model serves as a constitutive relation which relates the current density to the electric field, replacing Ohm's law which applies for a normal conductor, and it is used to specialize the general electric and magnetic fields that satisfy Maxwell's equations. In low-temperature superconductors where crystal boundaries offer little impediment to current flow the Bean model can be used quite generally, but the same is not necessarily true for high-temperature superconductors. Certainly the constitutive relation for typical BSCCO materials with only uniaxial crystal orientation, leading to wide angle grain boundaries, differs from that of a biaxially oriented YBCO coated conductor with only small angle grain boundaries. It is the latter for which the Bean model is best suited because it is more like a low-temperature superconductor in regard to the affect of grain boundaries. Use of the Bean model together with the solution of Maxwell's equations found here, leads to a result in close agreement with the loss expression obtained by Norris, at least for the case where I_0/I_c is approximately in the range of 0.5–1, where I_0 is the peak current and I_c the critical current. This range is the range of practical interest since within this range the loss varies by about a factor of one hundred, and at least within this range the Norris expression is confirmed.

2. Calculation of the general loss expression for transport current in a strip

The two most important attributes of a superconductor carrying transport current and obeying the Bean model are (a) for a peak transport current I_0 less than the critical current I_c a current-free and field-free region exists centered about the middle of the conductor and (b) current changes with time as a result of a moving boundary enclosing a cross-sectional area $A_1(t)$ (t the time) which changes the sign of the current density $J = \pm J_c$ as it moves (with J_c fixed in magnitude), as shown in Fig. 1. With each half-cycle of a cyclic state the boundary starts at the surface of the superconductor, of cross-sectional area A, and moves inward to a fixed boundary enclosing an area A_2, and then reforms around A for the next half-cycle. The area A_2 represents the minimum value of A_1 and it depends on the value of the peak current. The electric field which drives the current [13] results from a charge density on the surface of the superconductor that varies linearly along the length of a long uniform conductor, which has a similar parallel conductor as a return path. For a rectangular cross-section with the origin of the cross-sectional coordinates x, y at the middle of the conductor cross-section (current is along the z axis), and with the assumption that negligible true magnetization and electric polarization exist in the conductor, it was shown [14] that the precise electric field, including the effect of surface charge, is
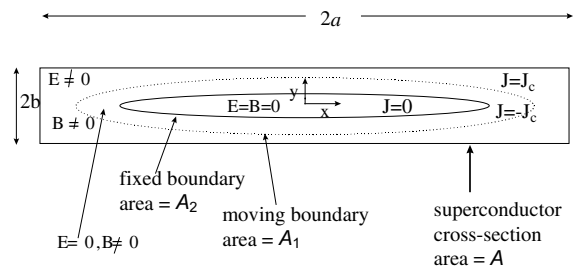
Fig. 1. Current and field distribution in the cross-section of a rectangular superconductor carrying transport current in the Bean model. In a cyclic state the current density in the Bean model can change only by means of a moving boundary which reverses the current density. The boundary starts on the surface of the superconductor at the beginning of a half-cycle that begins with negative peak current, and moves inward until it reaches a fixed boundary enclosing A_2 at the end of the half-cycle, corresponding to the case of positive peak current.

$$\mathbf{E}(x,y,t) = \hat{z}\frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int_A J(x',y',t)$$
$$\times \ln\frac{[(x'-x)^2+(y'-y)^2]}{[x'^2+y'^2]}\,dx'\,dy' \qquad (1)$$

with z along the length of the strip and the Bean model boundary condition $\mathbf{E}(0,0,t)=0$ imposed. The positive x axis is taken along the half-width a of the conductor and y along the half-thickness b. For the half-cycle of Fig. 1 the electric field for values of x, y between A and A_1 (where these values are denoted by >) is

$$\mathbf{E}_>(x,y,t) = -\hat{z}\frac{\mu_0}{2\pi}J_c\frac{\partial}{\partial t}$$
$$\times \int_{A_1} \ln\frac{[(x'-x)^2+(y'-y)^2]_>}{[x'^2+y'^2]}\,dx'\,dy' \qquad (2)$$

and if < denotes points x, y inside A_1

$$\mathbf{E}_<(x,y,t) = 0 \qquad (3)$$

which from continuity gives the boundary condition $\mathbf{E}_> = 0$ on the moving boundary. For the other half-cycle the current density has the opposite sign. The field-free region where both the electric and magnetic field, as well as the current density, are always zero is inside A_2. The flux density for the half-cycle of Fig. 1 is given by [14].

$$\mathbf{B}(x,y,t) = -\frac{\mu_0}{4\pi}J_c\left(\hat{x}\frac{\partial}{\partial y}-\hat{y}\frac{\partial}{\partial x}\right)$$
$$\times \int_{A+A_2-2A_1} \ln[(x'-x)^2+(y'-y)^2]\,dx'\,dy' \qquad (4)$$

where the integration is the sum of an integral over A and an integral over A_2 minus twice the integral over A_1. At the end of a half-cycle where $A_1 = A_2$ the integration is just over $A - A_2$. The loss per cycle Q per unit length ℓ is taken to be

$$\frac{Q}{\ell} = \oint dt \int_A dx\,dy\,\mathbf{J}\cdot\mathbf{E} \qquad (5)$$

where the closed integral indicates an integral over a time cycle. But since $\mathbf{J}$ vanishes inside A_2 the integral over A can be replaced with an integral over $A - A_2$. Then with an interchange of integrals

$$\frac{Q}{\ell} = \int_{A-A_2} dx\,dy \oint dt\,\mathbf{J}\cdot\mathbf{E}. \qquad (6)$$

For the half-cycle considered the current density above the moving boundary is $\mathbf{J} = \hat{z}J_c$ and consequently from (2)

$$(\mathbf{J}\cdot\mathbf{E})_> = -\frac{\mu_0}{2\pi}J_c^2\frac{\partial}{\partial t}$$
$$\times \int_{A_1} \ln\frac{[(x'-x)^2+(y'-y)^2]_>}{[x'^2+y'^2]}\,dx'\,dy' \qquad (7)$$

which is the same for both half-cycles since the current density now appears as J_c^2. When the integration is written as twice the value for a half-cycle

$$\oint dt\,\mathbf{J}\cdot\mathbf{E} = -\frac{\mu_0}{\pi}J_c^2\int_{\text{half-cycle}} dt\frac{\partial}{\partial t}$$
$$\times \int_{A_1} \ln\frac{[(x'-x)^2+(y'-y)^2]_>}{[x'^2+y'^2]}\,dx'\,dy'. \qquad (8)$$

But from (3) it follows that the boundary moves such that

$$\frac{\partial}{\partial t}\int_{A_1} \ln\frac{[(x'-x)^2+(y'-y)^2]_<}{[x'^2+y'^2]}\,dx'\,dy' = 0 \qquad (9)$$

and with the use of (9) Eq. (8) can be simplified to

$$\oint dt\,\mathbf{J}\cdot\mathbf{E} = -\frac{\mu_0}{\pi}J_c^2\int_{\text{half-cycle}} dt\frac{\partial}{\partial t}$$
$$\times \int_{A_1} \ln\frac{[(x'-x)^2+(y'-y)^2]}{[x'^2+y'^2]}\,dx'\,dy'. \qquad (10)$$

Since in the half-cycle A_1 is formed at A at the beginning and moves to A_2 the time integration gives

$$\oint dt\,\mathbf{J}\cdot\mathbf{E}$$
$$= -\frac{\mu_0}{\pi}J_c^2\left\{\int_{A_2} \ln\frac{[(x'-x)^2+(y'-y)^2]}{[x'^2+y'^2]}\,dx'\,dy'\right.$$
$$\left. - \int_A \ln\frac{[(x'-x)^2+(y'-y)^2]}{[x'^2+y'^2]}\,dx'\,dy'\right\} \qquad (11)$$

or more compactly

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$$\oint \mathrm{d}t\, \mathbf{J} \cdot \mathbf{E} = \frac{\mu_0}{\pi} J_c^2$$
$$\times \int_{A-A_2} \ln \frac{[(x'-x)^2 + (y'-y)^2]}{[x'^2 + y'^2]} \, \mathrm{d}x' \, \mathrm{d}y'. \tag{12}$$

Then the loss per cycle (6) becomes

$$\frac{Q}{\ell} = \frac{\mu_0}{\pi} J_c^2 \int_{A-A_2} \mathrm{d}x \, \mathrm{d}y$$
$$\times \int_{A-A_2} \ln \frac{[(x'-x)^2 + (y'-y)^2]}{[x'^2 + y'^2]} \, \mathrm{d}x' \, \mathrm{d}y'. \tag{13}$$

In this expression the only unknown is the shape of the boundary around A_2, and since the electric field vanishes on both sides of this boundary the boundary condition to be satisfied is provided by the magnetic field. It can be shown [14] that the shape of A_2 can be computed by demanding that $\mathbf{B}$ must vanish on A_2. With this boundary condition the result (13) holds for a rectangular strip of arbitrary thickness, and as pointed out to the author by Parker [15] it simply corresponds to a general expression given by Norris, although it was not explicitly written out. A straight-forward use of (13) involves first computing the boundary of A_2 which is then introduced into (13) to give the loss. However it is only the shape of A_2, not its value, which is difficult to compute. The value of A_2 in terms of the peak cyclic current I_0 is determined from $I_0 = J_c(A - A_2)$, and therefore

$$\frac{A_2}{A} = 1 - \frac{I_0}{I_c} \tag{14}$$

where $I_c = J_c A$ is the critical current.

3. Approximation for a coated superconductor

For a coated superconductor the thickness is a factor of 10^3–10^4 smaller than the width and for this reason the y dependence in the logarithm of (13) will be completely neglected to give

$$\frac{Q}{\ell} \approx \frac{\mu_0}{\pi} J_c^2 \int_{A-A_2} \mathrm{d}x \, \mathrm{d}y \int_{A-A_2} \ln \left(1 - \frac{x}{x'}\right)^2 \mathrm{d}x' \, \mathrm{d}y' \tag{15}$$

but still with the boundary condition $B = 0$ on the surface of A_2. Dependence of the logarithm on the y and y' coordinates is expected to be significant only in a very small volume near the edges of the strip, and the approximation (15) should apply except possibly when I_0/I_c approaches zero and A_2 approaches A, but in this case the loss is so small as to have little practical interest. For $A_2 = 0$ corresponding to $I_0 = I_c$ Eq. (15) gives

$$\frac{Q}{\ell} = \frac{\mu_0}{\pi} I_c^2 (2 \ln 2 - 1) \tag{16}$$

which is the value Norris obtained for this case.

4. Transformation of the loss expression for values of I_0/I_c near unity

Except for the case where I_0/I_c approaches zero it is useful to perform a transformation on (15). Let the integrals $\int_{A-A_2}$ be written as $\int_A - \int_{A_2}$. Then (15) becomes

$$\frac{Q}{\ell} = \frac{\mu_0}{\pi} J_c^2 \left[\int_A \mathrm{d}x \, \mathrm{d}y \int_A \ln \left(1 - \frac{x}{x'}\right)^2 \mathrm{d}x' \, \mathrm{d}y' \right.$$
$$- \int_A \mathrm{d}x \, \mathrm{d}y \int_{A_2} \ln \left(1 - \frac{x}{x'}\right)^2 \mathrm{d}x' \, \mathrm{d}y'$$
$$- \int_{A_2} \mathrm{d}x \, \mathrm{d}y \int_A \ln \left(1 - \frac{x}{x'}\right)^2 \mathrm{d}x' \, \mathrm{d}y'$$
$$\left. + \int_{A_2} \mathrm{d}x \, \mathrm{d}y \int_{A_2} \ln \left(1 - \frac{x}{x'}\right)^2 \mathrm{d}x' \, \mathrm{d}y' \right]. \tag{17}$$

A physical meaning that can be given to this transformation is that a fictitious uniform current density J_c is added and subtracted to the field-free region [1] of area A_2. The first set of integrals gives the loss for $I_0 = I_c$ and the last set gives the "loss" for the fictitious subtracted critical current in A_2. The middle two terms involve interactions. By interchanging the primes

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[1] However, to keep the correct boundary conditions on A_2 one must also imagine a fictitious applied magnetic field that is added and subtracted to the field-free region. On the surface of A_2 one of the imaginary applied fields cancels the field of the fictitious current of one sign and similarly for the other, leaving $B = 0$ on the surface due to all the fictitious currents and fields and the original boundary condition remains intact.

with the un-primes in the second term on the right and then interchanging the order of integration one can write

$$
\int_A dx\,dy \int_{A_2} \ln\left(1 - \frac{x}{x'}\right)^2 dx'\,dy
$$

$$
= \int_{A_2} dx\,dy \int_A \ln\left(1 - \frac{x'}{x}\right)^2 dx'\,dy' \tag{18}
$$

and performing all the integrations over A gives finally

$$
\frac{Q}{\ell} = \frac{\mu_0}{\pi} J_c^2 \Bigg[A^2 (2\ln 2 - 1)
$$

$$
- 2A \int_{A_2} dx\,dy \Bigg[\left(1 + \frac{x}{a}\right) \ln\left(1 + \frac{x}{a}\right)
$$

$$
+ \left(1 - \frac{x}{a}\right) \ln\left(1 - \frac{x}{a}\right) - \frac{1}{2}\ln\frac{x^2}{a^2} - 1 \Bigg]
$$

$$
+ \int_{A_2} dx\,dy \int_{A_2} \ln\left(1 - \frac{x}{x'}\right)^2 dx'\,dy' \Bigg]. \tag{19}
$$

With the definition

$$
G = -\frac{2}{A} \int_{A_2} dx\,dy \Bigg[\left(1 + \frac{x}{a}\right) \ln\left(1 + \frac{x}{a}\right) + \left(1 - \frac{x}{a}\right)
$$

$$
\times \ln\left(1 - \frac{x}{a}\right) - \frac{1}{2}\ln\frac{x^2}{a^2} - 1 \Bigg] \tag{20}
$$

one can write

$$
\frac{Q}{\ell} = \frac{\mu_0}{\pi} I_c^2 \Bigg\{ 2\ln 2 - 1 + G
$$

$$
+ \frac{1}{A^2} \int_{A_2} dx\,dy \int_{A_2} \ln\left(1 - \frac{x}{x'}\right)^2 dx'\,dy' \Bigg\}. \tag{21}
$$

For small values of A_2 (I_0/I_c approaching unity) the double integral over A_2 will be small for any boundary condition since it can be written as $k(A_2/A)^2$ or $k(1 - I_0/I_c)^2$ where k is a dimensionless factor. If k were evaluated for an isolated body of area A_2 which is elliptical in shape it would be equal to 0.5 according to Norris [6]. The latter leads to a small result for the double integral over A_2 but it is not completely negligible for the expansion to be considered below, which involves terms up to and including $(1 - I_0/I_c)^2$. But the proper

boundary condition $B = 0$ on each of the integrals over A_2 does not correspond to that for an isolated body, and it is expected that for the correct boundary condition (see the footnote) the double integral becomes much smaller, and one can simply ignore this term. Thus for small $(1 - I_0/I_c)$ one can write

$$
\frac{Q}{\ell} \approx \frac{\mu_0}{\pi} I_c^2 \{2\ln 2 - 1 + G\}. \tag{22}
$$

5. Evaluation of the loss for I_0/I_c in the neighborhood of unity

It was found [14] that for $0.46 < I_0/I_c < 1$ the field-free region was an ellipse-like figure, and for this range of values the height $y_2(x) = y_2(-x)$ of the boundary (Fig. 2) measured from the x axis was given by

$$
y_2(x) = b_2 \Bigg\{ 1 - \left[\frac{3}{5} - \frac{8}{25}\left(\frac{a_2}{a}\right)^2 + \cdots\right]\left(\frac{x}{a_2}\right)^2
$$

$$
- \left[\frac{1}{5} + \frac{16}{75}\left(\frac{a_2}{a}\right)^2 + \cdots\right]\left(\frac{x}{a_2}\right)^4
$$

$$
- \left[\frac{1}{5} + \frac{8}{75}\left(\frac{a_2}{a}\right)^2 + \cdots\right]\left(\frac{x}{a_2}\right)^6 \Bigg\}. \tag{23}
$$

The calculation was based on the use of an initial truncated series expansion in even powers of x/a_2 involving three arbitrary coefficients, where the coefficients were evaluated by demanding that A_2 must have the shape required to make the expression for **B**, given by (4) at the end of a half-cycle, vanish at points on the surface of A_2. The values obtained for the three coefficients are shown in (23) in the three sets of brackets and each is a series in pow-
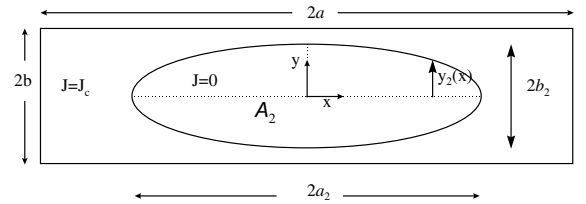

Fig. 2. A rectangular strip carrying a moderately large peak transport current I_0. The field-free region is ellipse-like where a_2 and b_2 are the principal axes. The detailed shape is described by $y_2(x)$.

ers of $(a_2/a)^2$, where a_2 is the principal axis of the ellipse-like figure along the x axis and a is the half-width of the superconducting strip. For I_0/I_c approaching unity $(a_2/a)^2$ is small compared with unity and the series is converging. The use of one or more additional coefficients multiplying higher powers of x/a_2 in (23) could slightly increase the accuracy of $y_2(x)$ but for I_0/I_c approaching unity it is believed that (23) is already quite accurate. It is observed that $y_2(x)$ meets the requirement that $y_2(a_2) = 0$ as it must for describing an ellipse-like figure. With the use of (23) the area of A_2 can be computed, giving with the aid of (14) the value of b_2/b in terms of I_0/I_c and $(a_2/a)^2$ where b_2 is the principal axis along the y direction, and b is the half-thickness of the strip. Finally, for small values of $(1 - I_0/I_c)$ an expression for $(a_2/a)^2$ was computed to be [14]

$$\left(\frac{a_2}{a}\right)^2 = 2.33\left(1 - \frac{I_0}{I_c}\right) - 1.08\left(1 - \frac{I_0}{I_c}\right)^2 + \cdots \tag{24}$$

With these results one can evaluate the expression for G which becomes

$$G = \left[\left(1 - \frac{I_0}{I_c}\right)\ln\left(1 - \frac{I_0}{I_c}\right) + 0.33\left(1 - \frac{I_0}{I_c}\right)\right.$$
$$\left. - 0.79\left(1 - \frac{I_0}{I_c}\right)^2 + \cdots\right] \tag{25}$$

with the next term in the expansion proportional to $(1 - I_0/I_c)^3$. Then (22) becomes

$$\frac{Q}{\ell} = \frac{\mu_0}{\pi}I_c^2\left[2\ln 2 - 1 + \left(1 - \frac{I_0}{I_c}\right)\ln\left(1 - \frac{I_0}{I_c}\right)\right.$$
$$\left. + 0.33\left(1 - \frac{I_0}{I_c}\right) - 0.79\left(1 - \frac{I_0}{I_c}\right)^2 + \cdots\right] \tag{26}$$

6. Comparison with Norris's result for I_0/I_c near unity

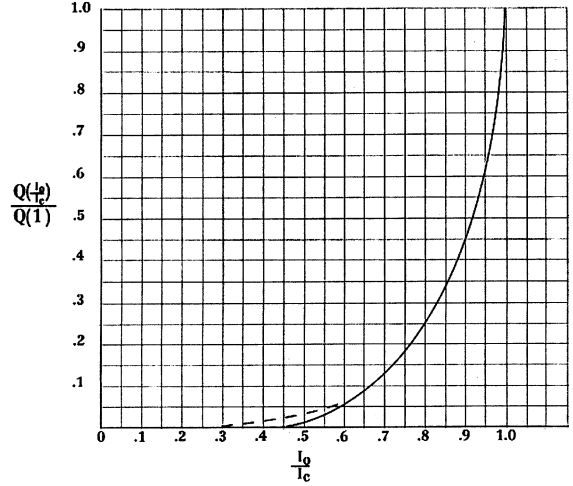The result given by Norris for a thin strip was

Fig. 3. Reduced plot of the loss versus the ratio of peak transport current to the critical current. The solid curve comes from the present calculations and the dashed curve from the Norris result. For I_0/I_c in the range of 0.46–1 the solid line, which comes from Eq. (26), falls on top of the Norris expression over most of the curve. For I_0/I_c less than 0.46 the precise loss values were not calculated except to show that they are very small. Consequently on the scale of the present plot they are shown as zero.

$$\left(\frac{Q}{\ell}\right)_{\text{Norris}} = \frac{\mu_0}{\pi}I_c^2\left[\left(1 - \frac{I_0}{I_c}\right)\ln\left(1 - \frac{I_0}{I_c}\right)\right.$$
$$\left. + \left(1 + \frac{I_0}{I_c}\right)\ln\left(1 + \frac{I_0}{I_c}\right) - \frac{I_0^2}{I_c^2}\right]. \tag{27}$$

A plot of (26) as shown in Fig. 3 falls closely on top of the Norris expression with deviations beginning to occur only near $I_0/I_c = 0.6$, and perhaps these small variations would be removed by a consideration of additional terms in the expansion. To understand why $2\ln 2 - 1 + G$ closely reproduces the Norris expression one can expand (27) in powers of $(1 - I_0/I_c)$ to obtain

$$\left(\frac{Q}{\ell}\right)_{\text{Norris}} = \frac{\mu_0}{\pi}I_c^2\left[2\ln 2 - 1 + \left(1 - \frac{I_0}{I_c}\right)\ln\left(1 - \frac{I_0}{I_c}\right)\right.$$
$$+ 0.307\left(1 - \frac{I_0}{I_c}\right) - 0.750\left(1 - \frac{I_0}{I_c}\right)^2$$
$$\left. + 0.042\left(1 - \frac{I_0}{I_c}\right)^3 + \cdots\right] \tag{28}$$

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which is essentially the same series as (26). Thus it is clear that for the range of current where A_2 is ellipse-like $(0.46 < I_0/I_c < 1)$ one can replace the expansion of (26) by the closed form of Norris and

$$\frac{Q}{\ell} \approx \frac{\mu_0}{\pi} I_c^2 \left[\left(1 - \frac{I_0}{I_c} \right) \ln \left(1 - \frac{I_0}{I_c} \right) \right.$$
$$\left. + \left(1 + \frac{I_0}{I_c} \right) \ln \left(1 + \frac{I_0}{I_c} \right) - \frac{I_0^2}{I_c^2} \right]. \qquad (29)$$

7. $B_y(x,y)$ for x in the field-free region

From $\operatorname{div} \mathbf{B} = 0$

$$\frac{\partial}{\partial y} B_y = -\frac{\partial}{\partial x} B_x \qquad (30)$$

and an integration over y gives

$$B_y(x,y) = B_y(x,0) - \int_0^y \mathrm{d}y \frac{\partial}{\partial x} B_x \qquad (31)$$

where $\partial B_x/\partial x$ is the order of B_x/a and the integral the order of $B_x b/a$ which for a coated superconductor is extremely small. Consequently $B_y(x,y) \approx B_y(x,0)$, as several other authors have also pointed out [11]. On the x axis within the field-free region (i.e. $-a_2 < x < a_2$), $B_y(x,0) = 0$ and therefore $B_y(x,y)$ is approximately zero over the entire thickness within this range.

8. The loss for small values of I_0/I_c

According to the three-constant truncated series expansion for $y(x)$ the value of a_2 becomes equal to a for $I_0/I_c < 0.46$ and the shape of the field-free region changes from ellipse-like to rectangular-like. Then from the approximation of the last section $B_y \approx 0$ over the whole strip and the loss all results from B_x, which according to Hancox [16,17] is very small for a thin strip. Thus, although the truncated series approximation does not give firm values for the loss in this region it does show the loss to be very small, and it is plotted as zero in Fig. 3. To obtain the actual result would require an infinite series expansion along with the fact that

a_2 is not precisely equal to a but deviates (because of edge effects) by at least a length equal to a few multiples of the strip thickness. The value of Norris's expression at $I_0/I_c = 0.46$ is down by about two powers of 10 from the value for $I_0/I_c = 1$ and therefore it also shows loss values near zero for the small I_0/I_c region. Consequently the extrapolation of the closed form of Norris to cover the complete range of current is probably as good an approximation as one can make. However, it is clear that at some value of I_0/I_c the Bean model should break down for thin strips due to surface effects. Surface effects which are ordinarily ignored in thick strips can become important for coated conductors. For example, the Bean current presumably changes over to a lossless London current at some small value of I_0/I_c.

9. Comparison with reported measurements on coated superconductors

Most of the reported measurements on coated superconductors disagree with the Norris expression for the dependence of loss on transport current in a thin rectangular strip, and therefore they disagree with the results obtained here. However, the authors of nearly all of the recently reported measurements have been concerned with non-uniformity of critical current in their samples. In particular Tsukamoto and Miyagi [3] have studied non-uniformity through the thickness, and Ashworth et al. [2] have studied variations in J_c along the length of the conductor. But it is noteworthy that in one sample measured by Iijima et al. [1] good agreement with the Norris expression was found over the wide range of I_0/I_c between 0.16 and 1. This sample had the higher J_c of the two samples measured by Iijima et al. but still the J_c was nearly an order of magnitude smaller than the J_c of most of the recently measured samples, which conceivably could have led to better uniformity. The fact that at least one sample has been found to agree with the theory tends to indicate that the theory is correct, and that disagreements which have been observed are due to non-uniformity in the measured samples.

10. Summary

In order to check the accuracy of previous loss theory for a "coated superconductor" carrying ac transport current in zero applied magnetic field a new method of computation is employed. It is assumed that the superconductor obeys the Bean model and Maxwell's equations (in the quasi-static approximation with neglect of true magnetization and electric polarization). The solution of Maxwell's equations for a given geometry is computed in terms of a vector and scalar potential, where these potentials are calculated in the usual way in terms of current density and charge density respectively. The electric field is $\mathbf{E} = -\dot{\mathbf{A}} - \nabla \phi$ where $\dot{\mathbf{A}}$ is the time derivative of the vector potential and ϕ is the scalar potential, and the Bean model is then introduced into this general expression to give the electric field needed for the loss calculation.

The calculation differs from the case of an applied magnetic field with zero applied transport current mainly in (a) the existence of a surface charge on the superconductor, which indicates that both a scalar potential and a vector potential are required to give the electric field, and (b) it is necessity to consider a return path for the transport current. Thus it is a three-dimensional problem and the loss depends on both the section of conductor being considered and its return path. The geometry for which the loss is computed assumes a long straight section of conductor with a similar parallel section as the return path, where the distance between conductors is large compared with the conductor cross-sectional dimensions, but small compared with the length of the sections.

Use of the Bean model creates a current and field-free region about the middle of the conductor cross-section, and this region depends upon the peak transport current. One novel feature of the calculation is that the shape of the field-free region used comes from an actual calculation (made by demanding that the magnetic field must vanish on this surface) rather than from a qualitative assumption.

For I_0/I_c within the range of 0.46–1 which is the range of principal interest the result obtained agrees closely with the result obtained by Norris, and therefore confirms Norris's loss expression, assuming the latter was meant to apply for the particular return path geometry considered here. No attempt was made here to calculate the loss for small values of $I_0/I_c < 0.46$ except to show that for these values the loss is extremely small and it was plotted as zero in Fig. 3. At some point, as I_0/I_c approaches zero it is likely that the Bean model no longer applies and a completely lossless London-type current should exist.

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